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Mh4714 Week 8

Week 8

Theorem 0.1

If f is differentiable over (a,b) and f has a maximum or minimum value at $c \in (a,b)$ then $f'(c) = 0$.

Proof

If f has a maximum at c then $f(x) \leq f(c), \forall x \in (a,b) \Rightarrow f(x) - f(c) \leq 0, \forall x \in (a,b)$.
Since $c \in (a,b)$ there is $x \in (a,b)$ with $x < c$ and $x > c$.
If $x < c$ then $\dfrac{f(x)-f(c)}{x-c} \geq 0 \Rightarrow \lim\limits_{x \to c^-} \dfrac{f(x)-f(c)}{x-c} \geq 0$.
If $x > c$ then $\dfrac{f(x)-f(c)}{x-c} \leq 0 \Rightarrow \lim\limits_{x \to c^+} \dfrac{f(x)-f(c)}{x-c} \leq 0$.
Therefore since, $\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c}$ exists we must have

$$\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} = 0 = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

i.e $f'(c) = 0$.

If f has a minimum at c then $f(x) \geq f(c), \forall x \in (a,b) \Rightarrow f(x) - f(c) \geq 0, \forall x \in (a,b)$.
Since $c \in (a,b)$ there is $x \in (a,b)$ with $x < c$ and $x > c$.
If $x < c$ then $\dfrac{f(x)-f(c)}{x-c} \leq 0 \Rightarrow \lim\limits_{x \to c^-} \dfrac{f(x)-f(c)}{x-c} \leq 0$.
If $x > c$ then $\dfrac{f(x)-f(c)}{x-c} \geq 0 \Rightarrow \lim\limits_{x \to c^+} \dfrac{f(x)-f(c)}{x-c} \geq 0$.

Therefore since, $\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ exists we must have

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}=0=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

i.e $f'(c)=0$.

□

We can combine this result with the theorems on boundedness and continuity to solve some max/min problems.

Example 0.2

Determine the maximum and minimum value respectively of $\dfrac{3x+4}{x^2+1}$ over $[-2,2]$. Justify your answers.

The function $f(x)=\dfrac{3x+4}{x^2+1}$ is continuous over $[-2,2]$ because $3x+4$ and x^2+1 are both polynomials and so are continuous at every point of $\mathbb{R}$ and $x^2+1\neq 0$. It follows then from the boundedness properties of continuous functions that $f(x)$ has a maximum and a minimum value in $[-2,2]$.

Now $f(x)=\dfrac{3x+4}{x^2+1}$ is also differentiable everywhere and so if a max or a min occurs at some point $c\in(-2,2)$ then, according to the theorem that we have just proved above, we must have $f'(c)=0$.
It follows then the max and min value either occur at an end-point of the interval or at some point $c\in(-2,2)$ such that $f'(c)=0$.

If we look for all points c in (-2,2) at which $f'(c)=0$ we can compare the values of f at these points and at the end-points in order to find the max and min values of $f(x)$.

$$f'(x)=\frac{3(x^2+1)-2x(3x+4)}{(x^2+1)^2}=\frac{-3x^2-8x+3}{(x^2+1)^2}=0\Rightarrow -3x^2-8x+3=0$$

$$\Rightarrow x=-3 \text{ or } x=\frac{1}{3}.$$

Therefore we compare the values $f(-2)=-0.4, f\left(\frac{1}{3}\right)=5.4, f(2)=2$.
We conclude then that 5.4 is the maximum value and -0.4 is the minimum value of f over [-2,2].